Entanglement requirements for implementing bipartite unitary operations

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We prove, using a new method based on map-state duality, lower bounds on entanglement resources needed to deterministically implement a bipartite unitary using separable (SEP) operations, which include LOCC (local operations and classical communication) as a particular case. It is known that the Schmidt rank of an entangled pure state resource cannot be less than the Schmidt rank of the unitary. We prove that if these ranks are equal the resource must be uniformly (maximally) entangled: equal nonzero Schmidt coefficients. Higher rank resources can have less entanglement: we have found numerical examples of Schmidt rank 2 unitaries which can be deterministically implemented, by either SEP or LOCC, using an entangled resource of two qutrits with less than one ebit of entanglement.

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I. INTRODUCTION

It is possible to carry out nonlocal quantum operations on multipartite systems using only local quantum operations and classical communications (LOCC) provided that the parties involved have access to a suitable entangled state, referred to as a resource. Given a large enough resource it is always possible to use teleportation to send all inputs to one party, who performs the operation and then distributes the results to the other parties using teleportation. In some cases it is possible to perform a nonlocal operation with less entanglement than is required by teleportation [1–6]. The question then arises as to how much entanglement is really necessary in order to implement a given nonlocal operation.

Our first result, that the Schmidt rank of the resource must be at least as great as that of the unitary [Theorem 1(a)], follows rather immediately from the fact that it is a separable (SEP) operation. This is analogous to the result given in [7] in which probabilistic (i.e. SLOCC) implementations are considered. Since SEP is contained in SLOCC, our Theorem 1(a) can be seen as a consequence of the result in [7], however we provide an independent proof along the way to our main result.

In contrast to the probabilistic case, the deterministic implementation of a unitary is only possible if the state meets certain entanglement requirements. For one thing, the entanglement of the resource must be at least as great as the entangling power of the unitary since entanglement cannot increase under SEP [8]. It has been shown that any deterministic controlled-unitary operator on two qubits implemented with bipartite LOCC using a resource of two entangled qubits necessarily requires a maximally entangled resource [9]. Our paper takes a different approach to the problem, using SEP, and provides a proof applicable to general unitaries of arbitrary dimension. We show that if the resource has Schmidt rank equal to that of the unitary, the resource must be uniformly entangled in the sense that all its nonzero Schmidt coefficients are the same [Theorem 1(b)]. These same restrictions apply to LOCC, as it is a particular case of SEP.

It is not hard to see that if the Schmidt rank of the resource is greater than the Schmidt rank of the unitary, then the resource need not be uniformly entangled (e.g. a larger rank resource that is majorized by a smaller rank maximally entangled state). We have found that it is in fact possible for such a larger rank resource to have less entanglement than would be required for a resource of Schmidt rank equal to that of the unitary. We have found examples of protocols in both SEP and LOCC which deterministically implement a controlled phase operation using less than one ebit of entanglement. In this case the unitary has Schmidt rank two and the resource has Schmidt rank three. Although the nonlocal unitary protocol given in [10] can with certain probability consume less than one ebit of entanglement [1], we believe that ours is the first example of carrying out such a protocol deterministically using less than one ebit of entanglement.

The remainder of this article is organized as follows. Section II sets up the problem of bipartite

[1] Although the protocol given in [10] is deterministic in the sense that it always succeeds in a finite number of steps, it is probabilistic in the amount of entanglement required. For any nontrivial unitary there is a chance that the protocol requires usage of the $|\psi_{\alpha_2}\rangle$ state, which has one ebit of entanglement. Thus, if the protocol only has access to a state with less than one ebit of entanglement there is a nonzero probability that the protocol cannot be carried out successfully.

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deterministic implementations of unitary operators using SEP. Section III provides the requisite background regarding map-state duality [11, 12] and atemporal diagrams [13–15]. Our main result is proved in Sec. IV using what we believe to be a new method based on the use of map-state duality. In Sec. V we consider the case of a resource of larger Schmidt rank. There is a brief conclusion in Sec. VI. An appendix details the implementation of a controlled unitary using a qutrit resource state of less than one ebit of entanglement.

II. NONLOCAL UNITARIES VIA SEPARABLE OPERATIONS

We are interested in carrying out a bipartite unitary map $U : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_{\bar{A}} \otimes \mathcal{H}_{\bar{B}}$, using as a resource an entangled state $|\psi\rangle$ on two ancillary systems $\mathcal{H}_a$ and $\mathcal{H}_b$, by means of a separable operation $\{E_k \otimes F_k\}$, $k = 1, 2, \ldots$. Here $E_k : \mathcal{H}_A \otimes \mathcal{H}_a \to \mathcal{H}_{\bar{A}}$ and $F_k : \mathcal{H}_B \otimes \mathcal{H}_b \to \mathcal{H}_{\bar{B}}$ together form a product Kraus operator. For U to be unitary it is necessary that the dimensions of the Hilbert spaces satisfy $d_A d_B = d_{\bar{A}} d_{\bar{B}}$, but we do not require that $d_A = d_{\bar{A}}$ or $d_A = d_B$. The separable operation must satisfy the usual closure condition [16]

$$\sum_k (E_k \otimes F_k)^\dagger (E_k \otimes F_k) = I_A \otimes I_a \otimes I_b \otimes I_B \quad (1)$$

which is depicted in Fig. 1(a).

In addition, for $|\Phi\rangle$ any pure input state on $\mathcal{H}_A \otimes \mathcal{H}_B$, the outcome of the operation will be a pure state

$$U\Big(|\Phi\rangle\langle\Phi|\Big)U^\dagger = \sum_k \Big(E_k \otimes F_k\Big)\Big(|\Phi\rangle\langle\Phi| \otimes |\psi\rangle\langle\psi|\Big)\Big(E_k \otimes F_k\Big)^\dagger \quad (2)$$

on $\mathcal{H}_{\bar{A}} \otimes \mathcal{H}_{\bar{B}}$. Since the protocol is assumed to be deterministic, every term on the right side is proportional to the same pure state and it must be the case that

$$(E_k \otimes F_k)\,|\psi\rangle = \alpha_k U, \quad (3)$$

with α_k some complex number. Note that both sides of (3) are operators acting on $\mathcal{H}_A \otimes \mathcal{H}_B$; Fig. 2(a) will help interpreting it correctly.

The resource $|\psi\rangle$ is assumed to have a Schmidt rank of D_ψ, which means it can be written in the form

$$|\psi\rangle = \sum_{i=1}^{D_\psi} \lambda_i\, |a_i\rangle \otimes |b_i\rangle\,. \quad (4)$$

for suitable orthonormal bases $\{|a_i\rangle\}$ and $\{|b_i\rangle\}$ of $\mathcal{H}_a$ and $\mathcal{H}_b$, with Schmidt coefficients $\lambda_i > 0$ for $i \leq D_\psi$.

Similarly, the bipartite operator U is assumed to have a Schmidt rank of D_U, meaning that it can be written in the form [17]

$$U = \sum_{i=1}^{D_U} \mu_i \mathcal{A}_i \otimes \mathcal{B}_i, \quad (5)$$

where $\{\mathcal{A}_i\}$ and $\{\mathcal{B}_i\}$ are bases of the operator spaces $\mathcal{L}(\mathcal{H}_A, \mathcal{H}_{\bar{A}},)$ and $\mathcal{L}(\mathcal{H}_B, \mathcal{H}_{\bar{B}},)$, orthonormal under the Frobenius (Hilbert-Schmidt) inner product, and $\mu_i > 0$ for $i \leq D_U$. Equivalently, D_U is the minimum number of terms needed in order to write U in the form $\sum \mathcal{C}_i \otimes \mathcal{D}_i$, without requiring $\mathcal{C}_i$ or $\mathcal{D}_i$ to be from an orthonormal basis.

III. MAP-STATE DUALITY AND DIAGRAMS

Map-state duality [11, 12] plays a central role in the proof that will follow. This is a general concept that is sometimes referred to as reshaping or a partial transpose [11] and in a specific manifestation is known as the Jamiołkowski or sometimes the Choi-Jamiołkowski isomorphism. States and maps are considered to both be tensors, and when a choice of orthonormal basis is fixed there is a natural linear relation between bras and kets (i.e. $|i\rangle \leftrightarrow \langle i|$ for all basis vectors $|i\rangle$) [2].

With this identification between bras and kets in place, the bipartite state $|\psi\rangle$ on the Hilbert space $\mathcal{H}_a \otimes \mathcal{H}_b$ can be identified with the linear map $\psi' : \mathcal{H}_b \to \mathcal{H}_a$ obtained by turning kets into bras on the $\mathcal{H}_b$ space:

$$|\psi\rangle = \sum_{ij} \psi_{ij}\, |a_i\rangle \otimes |b_j\rangle \to \psi' = \sum_{ij} \psi_{ij}\, |a_i\rangle\langle b_j| \quad (6)$$

Similarly, the operators U, E_k, and F_k, give rise to $U' : \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}} \to \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ (by turning bras into kets on $\mathcal{H}_A$ and kets into bras on $\mathcal{H}_{\bar{B}}$), $E'_k : \mathcal{H}_a \to \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ (by turning bras into kets on $\mathcal{H}_A$), and $F'^T_k : \mathcal{H}_B \otimes \mathcal{H}_{\bar{B}} \to \mathcal{H}_b$ (by turning bras into kets on

[2] It is also possible to formulate map-state duality in a basis independent manner [17], however this is not necessary for the present work.

$\mathcal{H}_b$ and kets into bras on $\mathcal{H}_{\bar{B}}$),

$$U' = \sum_{ijmn} \left\langle \bar{A}_j, \bar{B}_n | U | A_i, B_m \right\rangle \left| A_i, \bar{A}_j \right\rangle \left\langle B_m, \bar{B}_n \right| ,$$
$$E'_k = \sum_{ijm} \left\langle \bar{A}_j | E_k | A_i, a_m \right\rangle \left| A_i, \bar{A}_j \right\rangle \left\langle a_m \right| ,$$
$$F'^T_k = \sum_{ijm} \left\langle \bar{B}_j | F_k | B_i, b_m \right\rangle \left| b_m \right\rangle \left\langle B_i, \bar{B}_j \right| . \quad (7)$$

In the case of these three operators, *map-map duality* may be a more precise term, however we will use map-state duality to refer to any such partial transpose. The primed operator for F_k is denoted as F'^T_k in order to draw attention to the fact that its domain and range are swapped in comparison to E'_k.

The equations introduced so far make use of six distinct Hilbert spaces and tensors of various rank. In such situations the underlying structure of equations can be somewhat hidden when expressed using Dirac notation. Abstract index notation is more transparent but can become unwieldy. For this reason we provide atemporal diagrams, similar to those found in [13], which should aid the reader in following the arguments in the text.

Operators are designated by squares or rectangular boxes. As a matter of style, the state $|\psi\rangle$ and its corresponding operator ψ' will be represented as a circle instead of a square. Lines between these boxes represent tensor contraction, and these lines are labeled by the Hilbert spaces which they correspond to. Open lines on the left of a diagram represent the input to the total linear operator defined by the diagram, and open lines on the right represent outputs. Putting the inputs on the left means that operators are to be applied in a left-to-right manner, opposite to how algebraic equations are interpreted. As has been so far described, our diagrams are to be interpreted in exactly the same way as traditional quantum circuits as used for example in Nielsen and Chuang [18]. The primary difference between our diagrams and traditional circuits is that in the latter the horizontal direction is understood to represent the passage of time whereas our diagrams make no reference to time. The presence of a summation symbol has the obvious meaning: the linear operator depicted in the diagram denotes the terms of a series. The trace or partial trace operation is just a special case of tensor contraction and is denoted by joining the relevant spaces with a line. The identity operator is represented by a line. With minor changes in style our diagrams are equivalent to the atemporal diagrams of [13], and resemble other such schemes [14, 15].

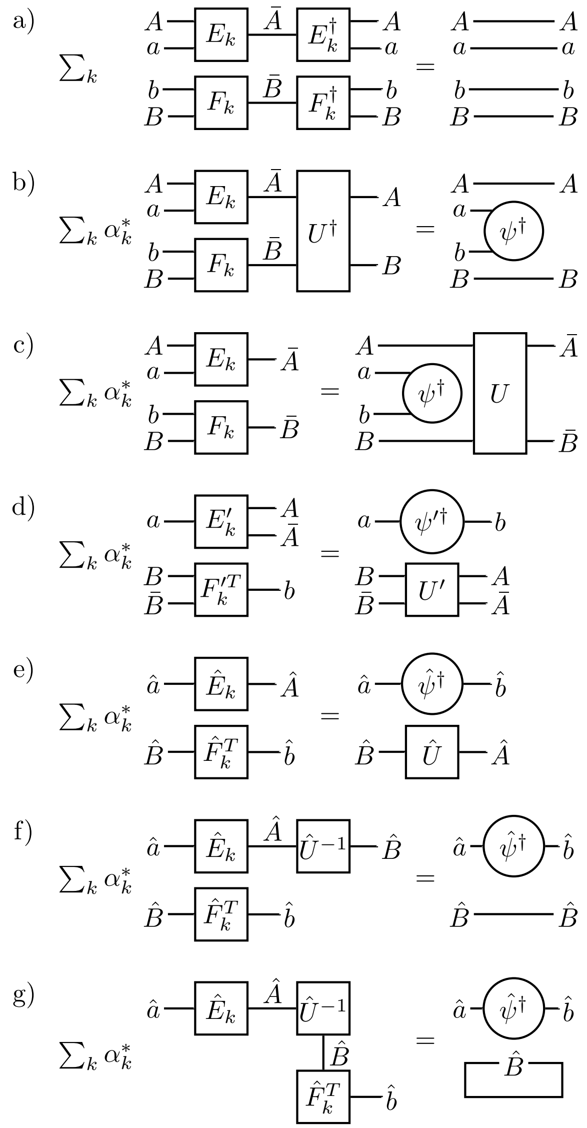

FIG. 1. Atemporal diagrams, explained in Sec. III. (a) Closure condition, (1). (b) Apply $\langle\psi|$ and simplify using the adjoint of Fig. 2(a) to get (9). (c) Multiply on the right by U to get (10). (d) Apply map-state duality to get (11). (e) Restrict spaces to supports and ranges of operators to get (12). (f) Multiply by $\hat{U}^{-1}$. (g) Trace over $\mathcal{H}_{\hat{B}}$ to get (13).

IV. ENTANGLEMENT REQUIREMENTS

Our main result is the following:

Theorem 1. *Suppose that a unitary operator U is implemented deterministically by a separable operation that makes use of the pure state entanglement resource $|\psi\rangle$ [i.e. suppose that* (1) *and* (3) *hold]. Then*

(a) The Schmidt rank D_ψ of $|\psi\rangle$ is greater than or equal to the Schmidt rank D_U of U.

(b) If the Schmidt ranks are equal, $D_U = D_\psi$, then

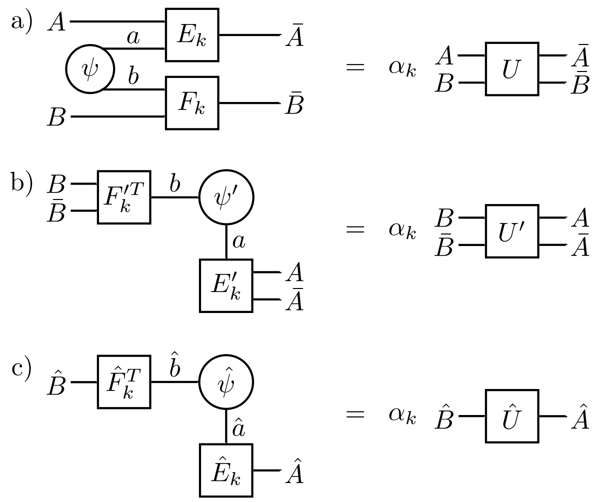

FIG. 2. (a) Deterministic unitary operation, (3). (b) Apply map-state duality to get (8). (c) Restrict spaces to supports and ranges of operators to get (14).

$|\psi\rangle$ must be a uniformly (maximally) entangled state: all the nonzero Schmidt coefficients are the same.

Proof of (a). Making use of map-state duality and the operators defined in (6) and (7), equation (3) [Fig. 2(a)] can be rewritten as [Fig. 2(b)]

$$E_k'\psi' F_k'^T = \alpha_k U'. \tag{8}$$

Since the rank of a product of linear operators is at most the smallest of the ranks of the individual operators, it follows that $\mathrm{rank}(\psi') \geq \mathrm{rank}(U')$. The rank of an operator is equal to the number of its nonzero singular values. Since the Schmidt decompositions (4) and (5) are essentially singular value decompositions of ψ' and U', it is apparent that $\mathrm{rank}(\psi') = D_\psi$ and $\mathrm{rank}(U') = D_U$ and the inequality becomes $D_\psi \geq D_U$. Part (a) is proved. □

Proof of (b). Apply the closure condition (1) to $\langle\psi|$ and use the adjoint of (3) to obtain

$$\sum_k \alpha_k^* U^\dagger (E_k \otimes F_k) = \langle\psi| \otimes I_{AB}. \tag{9}$$

as shown in Fig. 1(b). Next, multiply both sides on the left by U to arrive at

$$\sum_k \alpha_k^* (E_k \otimes F_k) = \langle\psi| \otimes U, \tag{10}$$

as shown in Fig. 1(c). Making use of map-state duality gives [Fig. 1(d)]

$$\sum_k \alpha_k^* (E_k' \otimes F_k'^T) = \psi'^\dagger \otimes U'. \tag{11}$$

The map U' may in general have rank less than the dimension of $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$ or $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ (which need not be equal to each other). In this case it will be useful to denote by $\mathcal{H}_{\hat{B}}$ the subspace of $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$ which forms the *support* (or co-image or row space) of U', the orthogonal complement of its kernel (null space), and by $\mathcal{H}_{\hat{A}}$ the subspace of $\mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ that forms the *range* (or image) of U'. Each of these subspaces has a dimension equal to D_U, and U' is a nonsingular (invertible) linear map of $\mathcal{H}_{\hat{B}}$ onto $\mathcal{H}_{\hat{A}}$, which we hereafter denote by $\hat{U}$. In the same way one can introduce subspaces $\mathcal{H}_{\hat{b}}$ and $\mathcal{H}_{\hat{a}}$ of $\mathcal{H}_b$ and $\mathcal{H}_a$ which form the support and range of ψ', and define $\hat{\psi}$ to be the corresponding nonsingular map of rank D_ψ from $\mathcal{H}_{\hat{b}}$ to $\mathcal{H}_{\hat{a}}$. Next, $\hat{E}_k$ is E_k' with its domain restricted to $\mathcal{H}_{\hat{a}}$, which can be strictly smaller than the support of E_k', and with its range restricted to $\mathcal{H}_{\hat{A}}$, which could be smaller than the image of E_k'. Finally, $\hat{F}_k^T$ is $F_k'^T$ regarded as a map from $\mathcal{H}_B \otimes \mathcal{H}_{\bar{B}}$ to $\mathcal{H}_b$, but with domain and range restricted to $\mathcal{H}_{\hat{B}}$ and $\mathcal{H}_{\hat{b}}$, respectively [3].

The result of restricting (11) to the subspaces just defined is

$$\sum_k \alpha_k^* (\hat{E}_k \otimes \hat{F}_k^T) = \hat{\psi}^\dagger \otimes \hat{U}, \tag{12}$$

corresponding to Fig. 1(e). Multiplying on the left by $\hat{U}^{-1}$ and tracing over $\mathcal{H}_{\hat{B}}$ gives

$$\sum_k \alpha_k^* \hat{F}_k^T \hat{U}^{-1} \hat{E}_k = D_U \hat{\psi}^\dagger, \tag{13}$$

see Fig. 1(f) and (g). Restricting (8) to subspaces results in

$$\hat{E}_k \hat{\psi} \hat{F}_k^T = \alpha_k \hat{U}, \tag{14}$$

see Fig. 2(c). Here we have restricted the spaces over which matrix multiplications are being performed ($\mathcal{H}_{\hat{b}}$ and $\mathcal{H}_{\hat{a}}$ instead of $\mathcal{H}_b$ and $\mathcal{H}_a$), however equality is still maintained because the dimensions which have been eliminated correspond to the zero Schmidt coefficients of $|\psi\rangle$, which is to say the zero singular values of ψ'.

To complete the proof, make use of the assumption $D_\psi = D_U$. Then D_ψ is also the rank of $\hat{E}_k$ and $\hat{F}_k^T$: all four operators in (14) are full rank. Taking the inverse of both sides and inserting the result for $\hat{U}^{-1}$ in (13) leads to the result

$$\sum_k |\alpha_k|^2 \hat{\psi}^{-1} = \hat{\psi}^{-1} = D_\psi \hat{\psi}^\dagger, \tag{15}$$

where $\sum_k |\alpha_k|^2 = 1$ follows from (1), (3) and the normalization of $|\psi\rangle$. With $|\psi\rangle$ in Schmidt form, $\hat{\psi}$ is

[3] It is significant that we define $\hat{F}_k^T$ as $F_k'^T$ restricted to subspaces. In general it is not the case that $\hat{F}_k$ is equal to F_k' restricted to subspaces.

diagonal, so $\hat{\psi} = I/\sqrt{D_\psi}$. Therefore all the nonzero Schmidt coefficients of $|\psi\rangle$ are equal to $1/\sqrt{D_\psi}$. $\square$

V. LARGER RANK RESOURCE

We have proved that a resource that is of the smallest viable Schmidt rank must be maximally entangled, but it is also possible to use a resource that is of higher Schmidt rank that is not maximally entangled. For one thing, if such a state meets an appropriate majorization criterion it can be deterministically transformed into a maximally entangled state [19]. In this case the larger rank initial resource would have greater entanglement than would be required if the smaller maximally entangled state had been used in the first place. There is however the possibility that some protocol could be devised to use a resource of larger Schmidt rank that has less entanglement than the maximally entangled state of smaller rank.

In fact, we have numerically found examples of such constructions in both SEP and LOCC. One solution in SEP uses a resource state $|\psi\rangle = \sqrt{0.81}\,|00\rangle + \sqrt{0.095}(|11\rangle + |22\rangle)$ on two qutrits to implement the two qubit controlled unitary operator $U = \mathrm{diag}\{1, 1, 1, e^{i\phi}\}$ with $\phi = 2\cos^{-1}(35/36)$. We have verified this to be an exact solution using a computer algebra system. This resource constitutes less than one ebit of entanglement: the Von Neumann entropy is approximately 0.89 ebits. Since entropy cannot increase under SEP [8] it is necessary for the resource that is consumed to have greater entanglement than the entangling capacity of the unitary being implemented. The entangling capacity of this unitary is shown in [20] to be approximately 0.23 ebits. Since this is much less than the 0.89 ebits that we use, there remains the possibility that a different construction or an even larger rank resource could potentially lower the entanglement cost further.

We also found an LOCC protocol which, though less efficient than the SEP construction just described, allows one to carry out a bipartite unitary deterministically using a resource with less than one ebit of entanglement. The resource in this case is $|\psi\rangle = \sqrt{0.8}\,|00\rangle + \sqrt{0.1}(|11\rangle + |22\rangle)$ and the unitary implemented is $U = \mathrm{diag}\{1, 1, 1, e^{i\phi}\}$ with $\phi = 0.08\pi$. The Von Neumann entropy of this resource is approximately 0.92 ebits, and this is a four round protocol (Alice, Bob, Alice, Bob).

The constructions described above are instances of a more general continuous family of solutions that we have found, covering a range of controlled phase operations. As should be expected, a larger phase ϕ requires a larger entanglement resource. In both the SEP and the LOCC case only certain classes of solutions were searched for, so it is possible that a more thorough search would provide more efficient protocols. The details of our SEP construction are presented in Appendix A. Our LOCC construction consists of a long list of Kraus operators in numerical form, which is available upon request.

VI. CONCLUSION

We have shown that a unitary operator of Schmidt rank D implemented as a bipartite separable operation requires an entanglement resource of Schmidt rank at least D. If the Schmidt rank of the resource is exactly equal to D, the resource must be uniformly (maximally) entangled with equal nonzero Schmidt coefficients. These restrictions apply also to LOCC, which is a subset of SEP. The proof uses map-state duality in a way which has not (so far as we know) been previously applied to problems of this type, so might have other interesting applications.

Numerical results show that the amount of entanglement required for the resource can be lowered by using a resource of Schmidt rank larger than D. A four round LOCC protocol has been found which uses a two-qutrit resource state with less than one ebit of entanglement to implement a bipartite controlled phase gate (albeit with a small phase).

Although some large classes of unitaries are known to have implementations in LOCC using resources having the minimal Schmidt rank required by Theorem 1(a) [1–4, 6], it is not known whether such minimal-rank implementations are possible for all unitaries. Given a unitary of Schmidt rank D_U it is always possible to find a collection of operators $\{E_k \otimes F_k\}$ such that (9) and (3) are satisfied with a resource of Schmidt rank $D_\psi = D_U$. But it is not known if there is a separable operation satisfying both (1) and (3). Consequently, it is possible that some unitaries may require a resource of greater rank than the lower bound given in Theorem 1(a). Even if such a minimal rank solution is always possible in SEP, it still might not be possible in LOCC. This stands in contrast to the case of SLOCC where it is known that any unitary can be implemented using a state of Schmidt rank equal to that of the unitary [7].

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Appendix A: Less than one ebit in SEP

We performed a numerical search for solutions to (1) and (3) with the resource and unitary taking the forms

$$|\psi\rangle = \sqrt{c_0}\,|00\rangle + \sqrt{(1-c_0)/2}(|11\rangle + |22\rangle), \quad \text{(A1)}$$
$$U = \text{diag}\{1,1,1,e^{i\theta}\}. \quad \text{(A2)}$$

In this case the unitary U is Schmidt rank 2 and the resource is Schmidt rank 3, so the spaces $\mathcal{H}_a$ and $\mathcal{H}_b$ are each 3 dimensional. In order to reduce the search space we looked for operators $\{E_k\}$ and $\{F_k\}$ of the form

$$E_k = E_* S_k \text{ and } F_k = F_* T_k \quad \text{(A3)}$$

where $S_k : \mathcal{H}_a \to \mathcal{H}_c$, $T_k : \mathcal{H}_b \to \mathcal{H}_c$ with $\mathcal{H}_c$ being a two dimensional space, and

$$E_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\bar{A}A} \otimes \langle 0|_c + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\bar{A}A} \otimes \langle 1|_c\,, \quad \text{(A4)}$$

$$F_* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\bar{B}B} \otimes \langle 0|_c + \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}_{\bar{B}B} \otimes \langle 1|_c\,. \quad \text{(A5)}$$

It is possible to take advantage of the symmetry of the resource $|\psi\rangle$ by searching for operator sets of the form

$$\{S_k L^l M^m N^n\} \text{ and } \{T_k L^l M^m N^n\} \quad \text{(A6)}$$

where $l, m, n \in \{0,1\}$ and L, M, and N are defined by

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{(A7)}$$

$M = \text{diag}\{1,1,-1\}$, and $N = \text{diag}\{1,-1,1\}$. There is no loss of generality in this assumption, since if $(\{S_k\},\{T_k\})$ gives a solution to (1) and (3) then so does $(\{\frac{1}{\sqrt{8}} S_k L^l M^m N^n\}, \{T_k L^l M^m N^n\})$. This decreases the number of independent operators (indexed by k) that need to be solved for, and in fact it turns out to be sufficient to consider only two values of k.

Initially we searched for solutions with $\theta = \pi/4$ and $c_0 = 0.6$ which, although representing more than one ebit of entanglement, is not majorized by a fully entangled resource of Schmidt rank 2. Once a solution was found, the parameters were variated until a value of c_0 was reached which represented a resource of less than one ebit of entanglement. Further constraints were added and variations made to simplify the solution and identify relations between the parameters. A family of solutions was found of the form (A6) with

$$S_0 = \begin{pmatrix} p & 1 & -p \\ e^{i\theta/2} & -p & -1 \end{pmatrix}, \quad \text{(A8)}$$

$$S_1 = \begin{pmatrix} -1 & 1 & -p \\ -pe^{i\theta/2} & p & 1 \end{pmatrix}, \quad \text{(A9)}$$

$$T_0 = \begin{pmatrix} -x-y & * & * \\ (x-y)e^{-i\theta/2} & * & * \end{pmatrix}, \quad \text{(A10)}$$

$$T_1 = \begin{pmatrix} -x+y & * & * \\ (x+y)e^{-i\theta/2} & * & * \end{pmatrix}, \quad \text{(A11)}$$

$$p = \sqrt{\frac{1-s}{s}}, \quad \text{(A12)}$$

$$s = x^2\,(1 - cos\,(\theta/2)) + y^2\,(1 + cos\,(\theta/2))\,, \quad \text{(A13)}$$

where the parameters x, y, c_0, and θ must be solved for numerically. The asterisks in T_0 and T_1 represent parameters that can be found using the relation $S_k \psi' T_k^T = I/4$.

A sequence of solutions for x, y, c_0, and θ were fed into an inverse symbolic calculator of our own design which uses a lookup table to convert floating point numbers into algebraic expressions. One of these solutions produced particularly simple algebraic expressions:

$$x = 9/5, \quad \text{(A14)}$$
$$y = -3/5, \quad \text{(A15)}$$
$$c_0 = 0.81, \quad \text{(A16)}$$
$$\theta = 2\text{arccos}(35/36). \quad \text{(A17)}$$

With this algebraic solution in hand, we used the computer algebra package Sage [21] to verify that this indeed represented an exact (not just approximate to within floating point precision) solution to (1) and (3).

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